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Generalized Stäckel transform and reciprocal transformations for finite-dimensional integrable systems

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Abstract
We present a multiparameter generalization of the Stäckel transform (the latter is also known as the coupling-constant metamorphosis) and show that under certain conditions this generalized Stäckel transform preserves Liouville integrability, noncommutative integrability and superintegrability. The corresponding transformation for the equations of motion proves to be nothing but a reciprocal transformation of a special form, and we investigate the properties of this reciprocal transformation. Finally, we show that the Hamiltonians of the systems possessing separation curves of apparently very different form can be related through a suitably chosen generalized Stäckel transform.

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1. Introduction

The Stäckel transform [12], also known as the coupling-constant metamorphosis [19], cf also [21–23, 36, 37] for more recent developments, is a powerful tool for producing new Liouville integrable systems from the known ones. This is essentially a transformation that sends an n-tuple of functions in involution on a $2n$-dimensional symplectic manifold into another n-tuple of functions on the same manifold, and these n new functions are again in involution. In its original form the Stäckel transform affects just one coupling constant which enters the Hamiltonian linearly and interchanges this constant with the energy eigenvalue (see [12, 19]).

In the present paper we introduce a multiparameter generalization of the classical Stäckel transform, which, just like its known counterpart, enables us to generate new Liouville integrable systems from the known ones or bring known integrable systems into a simpler form. Unlike the original Stäckel transform [12, 19] this *multiparameter generalized Stäckel* *transform* allows for the Hamiltonians being *nonlinear* functions of *several* parameters. These properties considerably increase the power of the transform in question.

Most importantly, under certain natural assumptions the multiparameter generalized Stäckel transform preserves Liouville integrability, superintegrability and noncommutative integrability, see propositions 1 and 2 and the related discussions.

Moreover, in section 4 we show that the transformations for equations of motion induced by the multiparameter generalized Stäckel transform are nothing but reciprocal transformations. This generalizes to the multiparameter case the earlier results of Hietarinta *et al* [19] on the one-parameter Stäckel transform.

The significance of reciprocal transformations in the theory of integrable nonlinear partial differential equations is well recognized. These transformations were intensively used in the theory of dispersionless (hydrodynamic-type) systems as well as in the theory of soliton systems, see e.g. [30, 32] and references therein. On the other hand, some particular examples of transformations of this kind for finite-dimensional Hamiltonian systems are also known, for instance the Jacobi transformation, see [25] and a recent paper [37]. The reciprocal transformations of somewhat different kind have also appeared in [19, 36, 39].

In the present paper we consider reciprocal transformations for the Liouville integrable Hamiltonian systems in conjunction with the generalized Stäckel transform and, in contrast with the earlier work on the subject, we concentrate on the multitime version of these transformations.

In fact, as we show in section 4, these transformations, when applied to the equations of motion of the source system, in general do *not* yield the equations of motion for the target system *unless* we restrict the equations of motion onto the common level surface of the corresponding Hamiltonians, see propositions 3 and 4 for details.

Thus, for two Liouville integrable systems related through a multiparameter generalized Stäckel transform for the constants of motion we have the reciprocal transformation relating the corresponding equations of motion restricted to appropriate Lagrangian submanifolds, see e.g. chapter 3 of [13] and references therein for more details on the latter.

Moreover, we present a multitime extension of the original reciprocal transformation from [19], and study the applications of this extended transformation to the integration of equations of motion in the Hamilton–Jacobi formalism using the separation of variables (cf [12]).

In the rest of the paper we consider classical Liouville integrable systems on $2n$-dimensional phase space. In [7] infinitely many classes of the Stäckel systems related to the so-called seed class, namely, the k-hole deformations of the latter, were constructed. Here we show that any k-hole deformation can be obtained from a seed-class system through a suitably chosen multiparameter generalized Stäckel transform, and present the explicit form of the transform in question along with its inverse.

2. Multiparameter generalized Stäckel transform: definition and duality

Let (M, P) be a Poisson manifold with the Poisson bracket $\{f, g\} = (\mathrm{d}f, P\,\mathrm{d}g)$. Consider r functionally independent Hamiltonians $H_i, i = 1, \ldots, r$, on M, and assume that these Hamiltonians further depend on $k \leqslant r$ parameters $\alpha_1, \ldots, \alpha_k$, so

$$H_i = H_i(x, \alpha_1, \ldots, \alpha_k), \qquad i = 1, \ldots, r, \tag{1}$$

where $x \in M$. Note that in general r is not related in any way to the dimension of M except for the obvious restriction $r \leqslant \dim M$; see, however, the discussion after proposition 1. Also, in what follows all functions will be tacitly assumed to be smooth (of the C^∞ class).

Suppose that there exists a k-tuple of pairwise distinct numbers $s_i \in \{1, \ldots, r\}$ such that

$$\det\left(\left\|\partial H_{s_i}/\partial\alpha_j\right\|_{i,j=1,\ldots,k}\right) \neq 0. \tag{2}$$

Now fix a k-tuple $\{s_1, \ldots, s_k\}$ such that (2) holds and consider the system

$$H_{s_i}(x, \alpha_1, \ldots, \alpha_k) = \tilde{\alpha}_i, \qquad i = 1, \ldots, k,$$

where $\tilde{\alpha}_i$ are arbitrary parameters, as a system of algebraic equations for $\alpha_1, \ldots, \alpha_k$. By the implicit function theorem, condition (2) guarantees that the solution of this system exists and is (locally) unique. We can write this solution in the form

$$\alpha_i = A_i(x, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k), \qquad i = 1, \ldots, k.$$

Now define the new Hamiltonians $\tilde{H}_{s_i}$, $i = 1, \ldots, k$, by setting

$$\tilde{H}_{s_i} = A_i(x, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k), \qquad i = 1, \ldots, k.$$

In other words, the Hamiltonians $\tilde{H}_{s_i}$, $i = 1, \ldots, k$, are defined by means of the relations

$$H_{s_i}|_{[\Phi]} = \tilde{\alpha}_i, \qquad i = 1, \ldots, k. \tag{3}$$

Here and below the subscript $[\Phi]$ means that we have substituted $\tilde{H}_{s_i}$ for α_i for all $i = 1, \ldots, k$.

Next, let

$$\tilde{H}_i = H_i|_{[\Phi]}, \qquad i = 1, \ldots, r, \qquad i \neq s_j \qquad \text{for} \quad j = 1, \ldots, k. \tag{4}$$

Note that the Hamiltonians $\tilde{H}_j$ involve k parameters $\tilde{\alpha}_i$, $i = 1, \ldots, k$, for all $j = 1, \ldots, r$:

$$\tilde{H}_i = \tilde{H}_i(x, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k), \qquad i = 1, \ldots, r.$$

We shall refer to the above transformation from H_i, $i = 1, \ldots, r$, to $\tilde{H}_i$, $i = 1, \ldots, r$, as to the k-parameter *generalized Stäckel transform* generated by $H_{s_1}, \ldots, H_{s_k}$. In analogy with [12] we shall say that the r-tuples H_i, $i = 1, \ldots, r$, and $\tilde{H}_i$, $i = 1, \ldots, r$, are *Stäckel-equivalent*.

The condition (2) guarantees that the above transformation is invertible. Indeed, consider the dual of the identity (3), that is,

$$\tilde{H}_{s_i}\big|_{[\tilde{\Phi}]} = \alpha_i, \qquad i = 1, \ldots, k, \tag{5}$$

where the subscript $[\tilde{\Phi}]$ means that we have substituted H_{s_i} for $\tilde{\alpha}_i$ for all $i = 1, \ldots, k$.

Moreover, the functional independence of the original Hamiltonians H_i, $i = 1, \ldots, r$, implies the functional independence of $\tilde{H}_i$, $i = 1, \ldots, r$. Indeed, the functional independence of H_i, $i = 1, \ldots, r$, means that $\dim \operatorname{span}(\mathrm{d}H_i, i = 1, \ldots, r) = r$ on an open dense subset U of M. Using (2), (3) and (4) we readily see that this implies $\dim \operatorname{span}(\mathrm{d}\tilde{H}_i, i = 1, \ldots, r) = r$ on another open dense subset $\tilde{U} \subset U$ of M. In turn, the latter equality means nothing but the functional independence of $\tilde{H}_i$, $i = 1, \ldots, r$ we sought for.

Let us stress that here and below the differentials are computed under the assumption that the parameters are considered to be constant, i.e., if $H = H(x, \alpha_1, \ldots, \alpha_k)$ then in the local coordinates x^b on M we have

$$\mathrm{d}H = \sum_{b=1}^{\dim M} \frac{\partial H}{\partial x^b}\,\mathrm{d}x^b.$$

By the implicit function theorem the condition (2) guarantees that we can solve (5) with respect to H_{s_j}, $j = 1, \ldots, k$. If we do this and define the remaining Hamiltonians H_i by the formulae

$$H_i = \tilde{H}_i|_{[\tilde{\Phi}]}, \qquad i = 1, \ldots, r, \qquad i \neq s_j \qquad \text{for} \quad j = 1, \ldots, k, \tag{6}$$

then it is straightforward to verify that (3) and (4) hold identically. In other words, formulae (5) and (6) define the inverse of the transformation defined using (3) and (4).

Clearly, these two transformations are dual, with the duality transformation swapping H_i and $\tilde{H}_i$ for all $i = 1, \ldots, r$ and swapping α_j and $\tilde{\alpha}_j$ for all $j = 1, \ldots, k$.

Note that in the special case when the Hamiltonians H_i are linear in the parameters α_j, the above formulae undergo considerable simplification, and we can explicitly express $\tilde{H}_i$ via H_i.

Namely, let

$$H_i = H_i^{(0)} + \sum_{j=1}^{k} \alpha_j H_i^{(j)}, \qquad i = 1, \ldots, r. \tag{7}$$

Then equations (3) take the form

$$H_{s_i}^{(0)} + \sum_{j=1}^{k} \tilde{H}_{s_j} H_{s_i}^{(j)} = \tilde{\alpha}_i, \qquad i = 1, \ldots, k, \tag{8}$$

and we can readily solve them for $\tilde{H}_{s_i}$:

$$\tilde{H}_{s_i} = \det W_i / \det W, \tag{9}$$

where W is a $k \times k$ matrix of the form

$$W = \left\| \begin{matrix} H_{s_1}^{(1)} & \cdots & H_{s_1}^{(k)} \\ \vdots & \ddots & \vdots \\ H_{s_k}^{(1)} & \cdots & H_{s_k}^{(k)} \end{matrix} \right\|,$$

and W_i are obtained from W by replacing $H_{s_j}^{(i)}$ by $\tilde{\alpha}_j - H_{s_j}^{(0)}$ for all $j = 1, \ldots, k$.

By (4) we have

$$\tilde{H}_i = H_i^{(0)} + \sum_{j=1}^{k} \tilde{H}_{s_j} H_i^{(j)}, \qquad i = 1, \ldots, r, \qquad i \neq s_j \qquad \text{for} \quad j = 1, \ldots, k, \tag{10}$$

where $\tilde{H}_{s_i}$ are given by (9). It is straightforward to verify that if we set $k = 1$ then the transformation given by (9) and (10) becomes nothing but the standard Stäckel transform [12], also known as the coupling-constant metamorphosis [19].

3. Multiparameter generalized Stäckel transform and (super)integrability

It turns out that the k-parametric generalized Stäckel transform preserves the commutativity of the Hamiltonians H_i. More precisely, we have the following result.

Proposition 1. *Let $H_i, i = 1, \ldots, r$, be functionally independent and let $\tilde{H}_i, i = 1, \ldots, r$, be related to $H_i, i = 1, \ldots, r$, by a k-parameter generalized Stäckel transform (3), (4) generated by $H_{s_1}, \ldots, H_{s_k}$, where $k \leqslant \operatorname{corank} P + (1/2) \operatorname{rank} P$.*

Then the following assertions hold:

(i) if $\{H_{s_i}, H_{s_j}\} = 0$ for all $i, j = 1, \ldots, k$ then $\{\tilde{H}_{s_i}, \tilde{H}_{s_j}\} = 0$ for all $i, j = 1, \ldots, k$;

(ii) under the assumptions of (i) suppose that $k + 1 \leqslant \operatorname{corank} P + (1/2) \operatorname{rank} P$ and for a $j_0 \in \{1, \ldots, r\}, j_0 \neq s_1, \ldots, s_k$ we have $\{H_{s_i}, H_{j_0}\} = 0$ for all $i = 1, \ldots, k$; then $\{\tilde{H}_{s_i}, \tilde{H}_{j_0}\} = 0$ for all $i = 1, \ldots, k$;

(iii) under the assumptions of (i) suppose that $k + 2 \leqslant \operatorname{corank} P + (1/2) \operatorname{rank} P$ and for $j_q \in \{1, \ldots, r\}, j_q \neq s_1, \ldots, s_k, q = 1, 2, j_1 \neq j_2$, we have $\{H_{s_i}, H_{j_q}\} = 0, i = 1, \ldots, k, q = 1, 2$, and $\{H_{j_1}, H_{j_2}\} = 0$; then $\{\tilde{H}_{s_i}, \tilde{H}_{j_q}\} = 0, i = 1, \ldots, k, q = 1, 2$, and $\{\tilde{H}_{j_1}, \tilde{H}_{j_2}\} = 0$.

Before we proceed with the proof of proposition 1, some remarks are in order. First, corank $P + (1/2)\,\mathrm{rank}\, P$ is easily seen to be the maximal possible number of functions in involution on M with respect to the Poisson bracket associated with P.

From proposition 1 it is immediate that the transformation defined by (3) and (4) preserves (super)integrability. Namely, under the assumptions of proposition 1 (i) let $\dim M = 2n$, $\mathrm{rank}\, P = 2n$, and consider r functionally independent Hamiltonians $H_i, i = 1, \ldots, r$, on M. Suppose that $\{H_{l_p}, H_{l_q}\} = 0$ for $p, q = 1, \ldots, m$, where $m \geqslant k$. Here $l_p \in \{1, \ldots, r\}$ are distinct integers such that $s_i \in \{l_1, \ldots, l_m\}$ for all $i = 1, \ldots, k$.

If $m = n$ then the dynamical system associated with any of H_{l_i} is Liouville integrable, as it has n commuting functionally independent integrals, $H_{l_j}, j = 1, \ldots, n$. By proposition 1 the dynamical system associated with any of $\tilde{H}_{l_i}$ enjoys the same property, the required integrals of motion in involution now being $\tilde{H}_{l_i}, i = 1, \ldots, n$.

If $m < n$ then, under some technical assumptions and in a suitable vicinity $U \subset M$, for the dynamical system associated with any of $H_{l_i}, i = 1, \ldots, m$, there exists a symplectic submanifold fibred into m-dimensional invariant tori [18, 28, 29]. The tori in question are intersections of this symplectic submanifold with the common level surfaces of $H_{l_i}, i = 1, \ldots, m$. Proposition 1 implies that this property is preserved by the multiparameter generalized Stäckel transform defined by (3) and (4), i.e., for the dynamical system associated with any of $\tilde{H}_{l_i}, i = 1, \ldots, m$, there exists, again under certain technical assumptions and in a suitable vicinity $\tilde{U} \subset M$, a symplectic submanifold fibred into m-dimensional invariant tori.

Now let us get back to the case of $m = n$ and further assume that $k < n, n < r \leqslant 2n - k$, and $\{H_{s_i}, H_j\} = 0$ for all $i = 1, \ldots, k$ and for all $j = 1, \ldots, r$. Note that the condition $r \leqslant 2n - k$ enables the relations $\{H_{s_i}, H_j\} = 0, i = 1, \ldots, k, j = 1, \ldots, r$, to hold without losing the functional independence of $H_i, i = 1, \ldots, r$, as the latter must hold by assumption.

Then the Hamiltonian H_{s_i} is superintegrable (see e.g. the survey [40] for the general definition of superintegrability) for any $i \in \{1, \ldots, k\}$ as it has $r > n$ integrals of motion $H_j, j = 1, \ldots, r$, and, moreover, there are n commuting integrals of motion $H_{l_p}, p = 1, \ldots, n$.

By proposition 1, (i)–(iii), the Hamiltonian $\tilde{H}_{s_j}$ is superintegrable for any $j \in \{1, \ldots, k\}$ as well, the integrals of motion now being $\tilde{H}_i, i = 1, \ldots, r$, and we have n *commuting* integrals of motion $\tilde{H}_{l_i}, i = 1, \ldots, n$. Thus, under certain technical assumptions the generalized Stäckel transform preserves superintegrability.

Moreover, the multiparameter generalized Stäckel transform defined by (3) and (4) also preserves noncommutative integrability in the sense of [9, 27]. Recall that a Hamiltonian dynamical system is said to be integrable in the noncommutative sense [9, 10, 27] if this system possesses an algebra of integrals of motion which is closed under the Poisson bracket and complete in the sense of [9, 10] (see also below). We start with the following result.

Proposition 2. *Under the assumptions of proposition 1 (i) suppose that* $\dim M = 2n$, P *is nondegenerate* ($\mathrm{rank}\, P = 2n$)*, and the algebra $\mathcal{F}$ of functions on M generated by $H_1, \ldots, H_r$ is closed under the Poisson bracket and is complete in the sense of [9]. Further suppose that* $\ker\{,\}|_{\mathcal{F}} = \mathcal{F}_0$*, where $\mathcal{F}_0$ is the algebra of functions on M generated by $H_{l_1}, \ldots, H_{l_m}, m \leqslant n$, where $l_j \in \{1, \ldots, r\}$, $j = 1, \ldots, m$, are distinct integers, H_{l_j}, $j = 1, \ldots, m$, are functionally independent, and $s_p \in \{l_1, \ldots, l_m\}$ for all $p = 1, \ldots, k$.*

Then the algebra $\tilde{\mathcal{F}}$ of functions on M generated by $\tilde{H}_1, \ldots, \tilde{H}_r$ is also closed under the Poisson bracket and complete.

Consider an algebra $\mathcal{A}$ of functions on a *symplectic* manifold M and assume that $\mathcal{A}$ is closed under the Poisson bracket. Recall (see [9] for precise definitions and further details) that the *differential dimension* $\operatorname{ddim}\mathcal{A}$ of $\mathcal{A}$ is, roughly speaking, the number of functionally independent generators of $\mathcal{A}$. The *differential index* $\operatorname{dind}\mathcal{A}$ can be (informally) defined as $\operatorname{dind}\mathcal{A} = \operatorname{ddim}\ker\{,\}|_{\mathcal{A}}$, and $\mathcal{A}$ is said to be *complete* [9, 10] if $\operatorname{ddim}\mathcal{A} + \operatorname{dind}\mathcal{A} = \dim M$ on an open dense subset $U \subset M$. Note that as H_i, $i = 1,\ldots,r$, are functionally independent generators of $\mathcal{F}$, we have $\operatorname{dind}\mathcal{F} = \operatorname{corank}||\{H_i, H_j\}||_{i,j=\overline{1,r}}$, see [10].

Sketch of proof of proposition 2. First, it is immediate that the algebra $\tilde{\mathcal{F}}$ generated by $\tilde{H}_1,\ldots,\tilde{H}_r$ is also closed under the Poisson bracket, so it remains to prove that $\tilde{\mathcal{F}}$ is complete.

As we have already noted in section 2, the functional independence of H_i, $i = 1,\ldots,r$, implies that of $\tilde{H}_i$, $i = 1,\ldots,r$, and hence we have $\operatorname{ddim}\tilde{\mathcal{F}} = \operatorname{ddim}\mathcal{F} = r$. In turn, as $\ker\{,\}|_{\mathcal{F}} = \mathcal{F}_0$, we have $\operatorname{dind}\mathcal{F} = \operatorname{ddim}\mathcal{F}_0 = m$.

Now, as $\mathcal{F}_0 = \ker\{,\}|_{\mathcal{F}}$ by assumption, we have $\left\{H_{l_i}, H_j\right\} = 0, i = 1,\ldots,m, j = 1,\ldots,r$. Hence by proposition 1 we obtain $\left\{\tilde{H}_{l_i}, \tilde{H}_j\right\} = 0, i = 1,\ldots,m, j = 1,\ldots,r$, and thus $\ker\{,\}|_{\tilde{\mathcal{F}}} \supset \tilde{\mathcal{F}}_0$, where $\tilde{\mathcal{F}}_0$ is the algebra of functions on M generated by $\tilde{H}_{l_1},\ldots,\tilde{H}_{l_m}$. Therefore $\operatorname{dind}\tilde{\mathcal{F}} \geqslant \operatorname{ddim}\tilde{\mathcal{F}}_0 = m$. However, we obviously have $\operatorname{ddim}\tilde{\mathcal{F}} + \operatorname{dind}\tilde{\mathcal{F}} \leqslant \dim M$ and, on the other hand, we know from the above that $\operatorname{ddim}\tilde{\mathcal{F}} + \operatorname{dind}\tilde{\mathcal{F}} \geqslant r + m = \dim M$. Hence $\operatorname{ddim}\tilde{\mathcal{F}} + \operatorname{dind}\tilde{\mathcal{F}} = \dim M$, and thus the algebra $\tilde{\mathcal{F}}$ is indeed complete. □

Therefore, if under the assumptions of proposition 2 for an integer $i_0 \in \{1,\ldots,r\}$ we have $\left\{H_{i_0}, H_j\right\} = 0$, $j = 1,\ldots,r$, and thus the dynamical system associated with H_{i_0} is completely integrable in the noncommutative sense [9, 10, 27], then so is the dynamical system associated with $\tilde{H}_{i_0}$.

Proof of proposition 1. Prove (i) first. For any smooth functions f and g on M that further depend on the parameters $\alpha_1,\ldots,\alpha_k$, we have the following easy identities:

$$\{f|_{[\Phi]}, g\}|_{[\Phi]} = \{f, g\}|_{[\Phi]} + \sum_{j=1}^{k}(\partial f/\partial\alpha_j)\big|_{[\Phi]}\big\{\tilde{H}_{s_j}, g\big\}\big|_{[\Phi]}, \tag{11}$$

$$\begin{aligned}\{f|_{[\Phi]}, g|_{[\Phi]}\} = \{f, g\}|_{[\Phi]} &+ \sum_{j=1}^{k}(\partial f/\partial\alpha_j)|_{[\Phi]}\big\{\tilde{H}_{s_j}, g\big\}|_{[\Phi]} + \sum_{j=1}^{k}(\partial g/\partial\alpha_j)|_{[\Phi]}\big\{f, \tilde{H}_{s_j}\big\}\big|_{[\Phi]}\\ &+ \sum_{i,j=1}^{k}(\partial f/\partial\alpha_i)|_{[\Phi]}(\partial g/\partial\alpha_j)|_{[\Phi]}\big\{\tilde{H}_{s_i}, \tilde{H}_{s_j}\big\}.\end{aligned} \tag{12}$$

Using the assumption $\left\{H_{s_i}, H_{s_j}\right\} = 0$ and (3), we find that

$$0 = \left\{\tilde{\alpha}_i - H_{s_i}, \tilde{\alpha}_j - H_{s_j}\right\} = \left\{H_{s_i}\big|_{[\Phi]} - H_{s_i}, H_{s_j}\big|_{[\Phi]} - H_{s_j}\right\},$$

whence

$$\left\{H_{s_i}\big|_{[\Phi]} - H_{s_i}, H_{s_j}\big|_{[\Phi]} - H_{s_j}\right\}\big|_{[\Phi]} = 0.$$

Writing the Poisson bracket on the left-hand side of the latter identity using (11) for the brackets $\left\{H_{s_i}\big|_{[\Phi]}, H_{s_j}\right\}\big|_{[\Phi]}$ and $\left\{H_{s_i}, H_{s_j}\big|_{[\Phi]}\right\}\big|_{[\Phi]}$, and (12) for the bracket $\left\{H_{s_i}\big|_{[\Phi]}, H_{s_j}\big|_{[\Phi]}\right\}$ we obtain

$$\sum_{p,q=1}^{k}\left(\partial H_{s_i}/\partial\alpha_p\right)\big|_{[\Phi]}\left(\partial H_{s_j}/\partial\alpha_q\right)\big|_{[\Phi]}\big\{\tilde{H}_{s_p}, \tilde{H}_{s_q}\big\}\big|_{[\Phi]} = 0,$$

whence using (2) we readily find that for all $p, q = 1,\ldots,k$ we have

$$\big\{\tilde{H}_{s_p}, \tilde{H}_{s_q}\big\}\big|_{[\Phi]} = 0.$$

However, $\tilde{H}_{s_p}$ are independent of α_j for all $j = 1, \ldots, k$, so

$$\{\tilde{H}_{s_p}, \tilde{H}_{s_q}\} = \{\tilde{H}_{s_p}, \tilde{H}_{s_q}\}|_{[\Phi]} = 0,$$

and the result follows.

As we have already proved (i), to prove (ii) we only need to show that if $\{H_{s_i}, H_{j_0}\} = 0$ for all $i = 1, \ldots, k$ then $\{\tilde{H}_{s_i}, \tilde{H}_{j_0}\} = 0$ for all $i = 1, \ldots, k$.

As $\tilde{H}_i$, $i = 1, \ldots, r$, are independent of α_p for all $p = 1, \ldots, k$ by construction, we have

$$\{\tilde{H}_{s_i}, \tilde{H}_{j_0}\} = \{\tilde{H}_{s_i}, \tilde{H}_{j_0}\}|_{[\Phi]}.$$

Moreover, as $j_0 \neq s_p$ for all $p = 1, \ldots, k$ by assumption, by virtue of (4) the relation $\{\tilde{H}_{s_i}, \tilde{H}_{j_0}\}|_{[\Phi]} = 0$ is equivalent to

$$\{\tilde{H}_{s_i}, H_{j_0}|_{[\Phi]}\}|_{[\Phi]} = 0.$$

In turn, using (11) we can rewrite the Poisson bracket $\{\tilde{H}_{s_i}, H_{j_0}|_{[\Phi]}\}|_{[\Phi]}$ as follows:

$$\{\tilde{H}_{s_i}, H_{j_0}|_{[\Phi]}\}|_{[\Phi]} = \{\tilde{H}_{s_i}, H_{j_0}\}|_{[\Phi]} - \sum_{p=1}^{k} (\partial H_{j_0}/\partial \alpha_p)|_{[\Phi]} \{\tilde{H}_{s_p}, \tilde{H}_{s_i}\}|_{[\Phi]}.$$

As $\{\tilde{H}_{s_p}, \tilde{H}_{s_i}\} = 0$ by (i), we see that

$$\{\tilde{H}_{s_i}, H_{j_0}|_{[\Phi]}\}|_{[\Phi]} = \{\tilde{H}_{s_i}, H_{j_0}\}|_{[\Phi]}.$$

Now, in analogy with the proof of (i), consider the identity

$$0 = \{\tilde{\alpha}_p, H_{j_0}\}|_{[\Phi]} = \{H_{s_p}|_{[\Phi]}, H_{j_0}\}|_{[\Phi]}.$$

Using (11) and our assumptions yields

$$0 = \{H_{s_p}|_{[\Phi]}, H_{j_0}\}|_{[\Phi]} = \sum_{i=1}^{k} (\partial H_{s_p}/\partial \alpha_i)|_{[\Phi]} \{\tilde{H}_{s_i}, H_{j_0}\}|_{[\Phi]}.$$

Finally, using (2) we conclude that

$$\{\tilde{H}_{s_i}, H_{j_0}\}|_{[\Phi]} = 0, \tag{13}$$

whence $\{\tilde{H}_{s_i}, H_{j_0}|_{[\Phi]}\}|_{[\Phi]} = 0$, and the result follows.

Part (iii) is proved in analogy with (ii). Namely, in view of (i) and (ii) we only need to prove that $\{H_{j_1}, H_{j_2}\} = 0$ implies $\{\tilde{H}_{j_1}, \tilde{H}_{j_2}\} = 0$ provided $j_q \neq s_p$ for all $p = 1, \ldots, k$ and $q = 1, 2$.

Then we have

$$\{\tilde{H}_{j_1}, \tilde{H}_{j_2}\} = \{H_{j_1}|_{[\Phi]}, H_{j_2}|_{[\Phi]}\}.$$

Using (13) for $j_0 = j_1$ and $j_0 = j_2$, and (12), we readily find that

$$\{H_{j_1}|_{[\Phi]}, H_{j_2}|_{[\Phi]}\} = 0,$$

and the result follows. □

Note that the computations in the above proof bear considerable resemblance to those in the theory of Hamiltonian systems with second-class constraints, see e.g. the classical book of Dirac [15].

4. Reciprocal transformations for the equations of motion

Recall that the equations of motion associated with a Hamiltonian H and a Poisson structure P on M read (see e.g. [3])

$$\mathrm{d}x^b/\mathrm{d}t_H = (X_H)^b, \qquad b = 1, \ldots, \dim M, \tag{14}$$

where x^b are local coordinates on M, $X_H = P\,\mathrm{d}H$ is the Hamiltonian vector field associated with H and t_H is the corresponding evolution parameter (time).

Throughout the rest of this section we tacitly assume that $\tilde{H}_i, i = 1, \ldots, r$, are related to $H_i, i = 1, \ldots, r$, through the k-parameter Stäckel transform (3), (4) generated by $H_{s_1}, \ldots, H_{s_k}$.

Suppose that $\{H_{s_i}, H_{s_j}\} = 0$ for all $i, j = 1, \ldots, k$, and consider simultaneously the equations of motion (14) for the Hamiltonians H_{s_i} with the times t_{s_i} and for $\tilde{H}_{s_i}$ with the times $\tilde{t}_{s_i}$:

$$\mathrm{d}x^b/\mathrm{d}t_{s_i} = \left(X_{H_{s_i}}\right)^b, \qquad b = 1, \ldots, \dim M, \qquad i = 1, \ldots, k, \tag{15}$$

$$\mathrm{d}x^b/\mathrm{d}\tilde{t}_{s_i} = \left(X_{\tilde{H}_{s_i}}\right)^b, \qquad b = 1, \ldots, \dim M, \qquad i = 1, \ldots, k. \tag{16}$$

In analogy with [19] consider a *reciprocal transformation* (see e.g. [30, 32, 33] for general information on such transformations) relating the times t_{s_i} and $\tilde{t}_{s_j}$:

$$\mathrm{d}\tilde{t}_{s_i} = -\sum_{j=1}^{k} \left.\left(\frac{\partial H_{s_j}}{\partial \alpha_i}\right)\right|_{[\Phi]} \mathrm{d}t_{s_j}, \qquad i = 1, \ldots, k. \tag{17}$$

Proposition 3. *Suppose that* $k \leqslant \operatorname{corank} P + (1/2) \operatorname{rank} P$ *and* $\{H_{s_i}, H_{s_j}\} = 0$ *for all* $i, j = 1, \ldots, k$, *and consider the equations of motion (15) for* $H_{s_i}, i = 1, \ldots, k$, *restricted onto the common level surface* $N_{\tilde{\alpha}}$ *of* H_{s_i}, *where*

$$N_{\tilde{\alpha}} = \left\{x \in M \,\middle|\, H_{s_i}(x, \alpha_1, \ldots, \alpha_k) = \tilde{\alpha}_i, i = 1, \ldots, k\right\}.$$

Then the transformation (17) is well defined on these restricted equations of motion and sends them into the equations of motion (16) for $\tilde{H}_{s_i}, i = 1, \ldots, k$, *restricted onto the common level surface* $\tilde{N}_\alpha$ *of* $\tilde{H}_{s_i}$, *where*

$$\tilde{N}_\alpha = \left\{x \in M | \tilde{H}_{s_i}(x, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) = \alpha_i, i = 1, \ldots, k\right\}.$$

Note that the level surfaces in question, $\tilde{N}_\alpha$ and $N_{\tilde{\alpha}}$, represent the *same* submanifold of M, i.e., $\tilde{N}_\alpha = N_{\tilde{\alpha}}$. This is readily verified using the relations (3) and (5).

Proof. First show that (17) is well defined, that is, we have

$$\frac{\partial^2 \tilde{t}_{s_i}}{\partial t_{s_p} \partial t_{s_q}} = \frac{\partial^2 \tilde{t}_{s_i}}{\partial t_{s_q} \partial t_{s_p}}, \qquad p, q = 1, \ldots, k, \tag{18}$$

by virtue of equations (15) restricted onto $N_{\tilde{\alpha}}$.

Using (17) we find that (18) boils down to

$$\left.\left(\frac{\partial \left(\frac{\partial H_{s_p}}{\partial \alpha_i}\right)\big|_{[\Phi]}}{\partial t_{s_q}}\right)\right|_{N_{\tilde{\alpha}}} = \left.\left(\frac{\partial \left(\frac{\partial H_{s_q}}{\partial \alpha_i}\right)\big|_{[\Phi]}}{\partial t_{s_p}}\right)\right|_{N_{\tilde{\alpha}}}, \qquad p, q = 1, \ldots, k. \tag{19}$$

In turn, using (15) we readily find that (19) takes the form

$$\left.\left\{\left.\left(\frac{\partial H_{s_p}}{\partial \alpha_i}\right)\right|_{[\Phi]}, H_{s_q}\right\}\right|_{N_{\tilde{\alpha}}} = \left.\left\{\left.\left(\frac{\partial H_{s_q}}{\partial \alpha_i}\right)\right|_{[\Phi]}, H_{s_p}\right\}\right|_{N_{\tilde{\alpha}}},$$

and the latter equality can be proved by taking the partial derivative of the relation $\{H_{s_p}, H_{s_q}\} = 0$ with respect to α_i.

Next, equation (17) yields

$$\frac{\mathrm{d}}{\mathrm{d}t_{s_i}} = -\sum_{j=1}^{k} \left(\frac{\partial H_{s_i}}{\partial \alpha_j}\right)\bigg|_{[\Phi]} \frac{\mathrm{d}}{\mathrm{d}\tilde{t}_{s_j}}, \qquad i = 1, \ldots, k.$$

Taking into account (15) and (16) we conclude that we have to prove that

$$X_{H_{s_i}}\big|_{N_{\tilde{\alpha}}} = -\sum_{j=1}^{k} \left(\left(\frac{\partial H_{s_i}}{\partial \alpha_j}\right)\bigg|_{[\Phi]}\right)\bigg|_{N_{\tilde{\alpha}}} X_{\tilde{H}_{s_j}}\big|_{N_{\tilde{\alpha}}}, \qquad i = 1, \ldots, k, \tag{20}$$

where $|_{N_{\tilde{\alpha}}}$ denotes restriction onto $N_{\tilde{\alpha}}$.

As $X_H = P\,\mathrm{d}H$ for any smooth function H on M, equation (20) boils down to

$$\left(P\left(\mathrm{d}H_{s_i} + \sum_{j=1}^{k} \left(\frac{\partial H_{s_i}}{\partial \alpha_j}\right)\bigg|_{[\Phi]} d\tilde{H}_{s_j}\right)\right)\bigg|_{N_{\tilde{\alpha}}} = 0, \qquad i = 1, \ldots, k. \tag{21}$$

On the other hand, taking the differential of (3) we obtain

$$(\mathrm{d}H_{s_i})|_{[\Phi]} + \sum_{j=1}^{k} \left(\frac{\partial H_{s_i}}{\partial \alpha_j}\right)\bigg|_{[\Phi]} (\mathrm{d}\tilde{H}_{s_j})|_{[\Phi]} = 0, \qquad i = 1, \ldots, k. \tag{22}$$

As $\tilde{H}_{s_j}$ are independent of α_p, for all $p = 1, \ldots, k$ we have $(\mathrm{d}\tilde{H}_{s_j})\big|_{[\Phi]} = \mathrm{d}\tilde{H}_{s_j}$, so (22) yields

$$\sum_{j=1}^{k} \left(\frac{\partial H_{s_i}}{\partial \alpha_j}\right)\bigg|_{[\Phi]} \mathrm{d}\tilde{H}_{s_j} = -(\mathrm{d}H_{s_i})|_{[\Phi]},$$

and (20) takes the form

$$\left(P\left(\mathrm{d}H_{s_i} - (\mathrm{d}H_{s_i})|_{[\Phi]}\right)\right)\big|_{N_{\tilde{\alpha}}} = 0, \qquad i = 1, \ldots, k.$$

In the local coordinates x^b on M we have

$$\begin{aligned}\left(P\left(\mathrm{d}H_{s_i} - (\mathrm{d}H_{s_i})|_{[\Phi]}\right)\right)\big|_{N_{\tilde{\alpha}}} &= \left(P\left(\sum_{b=1}^{\dim M}\left(\frac{\partial H_{s_i}}{\partial x^b} - \left(\frac{\partial H_{s_i}}{\partial x^b}\right)\bigg|_{[\Phi]}\right)\mathrm{d}x^b\right)\right)\bigg|_{N_{\tilde{\alpha}}} \\ &= \sum_{b=1}^{\dim M}\left(\frac{\partial H_{s_i}}{\partial x^b} - \left(\frac{\partial H_{s_i}}{\partial x^b}\right)\bigg|_{[\Phi]}\right)\bigg|_{N_{\tilde{\alpha}}} (P\,\mathrm{d}x^b)|_{N_{\tilde{\alpha}}}, \qquad i = 1, \ldots, k. \end{aligned}\tag{23}$$

By virtue of (3) and (5) $N_{\tilde{\alpha}}$ and $\tilde{N}_\alpha$ represent the same submanifold of M, whence

$$\left(\frac{\partial H_{s_i}}{\partial x^b} - \left(\frac{\partial H_{s_i}}{\partial x^b}\right)\bigg|_{[\Phi]}\right)\bigg|_{N_{\tilde{\alpha}}} = \left(\frac{\partial H_{s_i}}{\partial x^b} - \left(\frac{\partial H_{s_i}}{\partial x^b}\right)\bigg|_{[\Phi]}\right)\bigg|_{\tilde{N}_\alpha} = \left(\frac{\partial H_{s_i}}{\partial x^b}\right)\bigg|_{\tilde{N}_\alpha} - \left(\frac{\partial H_{s_i}}{\partial x^b}\right)\bigg|_{\tilde{N}_\alpha} = 0.$$

We used here an easy identity

$$\left(\left(\frac{\partial H_{s_i}}{\partial x^b}\right)\bigg|_{[\Phi]}\right)\bigg|_{\tilde{N}_\alpha} = \left(\frac{\partial H_{s_i}}{\partial x^b}\right)\bigg|_{\tilde{N}_\alpha}.$$

Thus, the left-hand side of (23), and therefore that of (21), vanishes, and the result follows. □

Now assume that all H_i are in involution:

$$\{H_i, H_j\} = 0, \qquad i, j = 1, \ldots, r.$$

Then by proposition 1 so are $\tilde{H}_i$, i.e.,

$$\{\tilde{H}_i, \tilde{H}_j\} = 0, \qquad i, j = 1, \ldots, r,$$

and we can consider two sets of simultaneous evolutions,

$$\mathrm{d}x^b/\mathrm{d}t_i = \left(X_{H_i}\right)^b, \qquad b = 1, \ldots, \dim M, \qquad i = 1, \ldots, r, \tag{24}$$

$$\mathrm{d}x^b/\mathrm{d}\tilde{t}_i = \left(X_{\tilde{H}_i}\right)^b, \qquad b = 1, \ldots, \dim M, \qquad i = 1, \ldots, r, \tag{25}$$

and the following extension of (17):

$$\begin{aligned} \mathrm{d}\tilde{t}_{s_i} &= -\sum_{j=1}^{r} \left(\frac{\partial H_j}{\partial \alpha_i}\right)\Bigg|_{[\Phi]} \mathrm{d}t_j, \qquad i = 1, \ldots, k, \\ \tilde{t}_q &= t_q, \qquad q = 1, 2, \ldots, r, \qquad q \neq s_p \quad \text{for any} \quad p = 1, \ldots, k. \end{aligned} \tag{26}$$

In analogy with proposition 3 we can prove the following result.

Proposition 4. *Suppose that* $\{H_i, H_j\} = 0$ *for all* $i, j = 1, \ldots, r$ *and* $r \leqslant \operatorname{corank} P + (1/2)\operatorname{rank} P$*, and consider the equations of motion (24) for* H_i, $i = 1, \ldots, r$*, restricted onto* $N_{\tilde{\alpha}}$*.*

Then the transformation (26) is well defined on these restricted equations of motion and sends them into the equations of motion (25) for $\tilde{H}_i$, $i = 1, \ldots, r$*, restricted onto* $\tilde{N}_\alpha$*.*

Note that the transformations from propositions 3 and 4 do not change the dynamical variables x. In particular, under the assumptions of proposition 3 for any given i from 1 to k the trajectories of the dynamical system associated with H_{s_i} are identical to those of the dynamical system associated with $\tilde{H}_{s_i}$, if we consider the trajectories as *non-parametrized* curves. In other words, the transformation (17) amounts to the reparametrization of the times associated with H_{s_j} for all $j = 1, \ldots, k$. Note, however, that in general the reparametrization in question is *different* for different trajectories, as one can readily infer from (17).

As a final remark note that it could be interesting to compare the above reparametrization results with those arising in the theory of projectively equivalent metrics [11, 35].

5. Canonical Poisson structure

In this section we tacitly assume that $\tilde{H}_i$, $i = 1, \ldots, r$, are related to H_i, $i = 1, \ldots, r$, through the k-parameter Stäckel transform (3), (4) generated by $H_{s_1}, \ldots, H_{s_k}$. We further assume that $M = \mathbb{R}^{2n}$, P is a canonical Poisson structure on M, and λ_i, μ_i, $i = 1, \ldots, n$, are the Darboux coordinates for P, i.e., $\{\lambda_i, \mu_j\} = \delta_{ij}$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$. Then the Hamilton–Jacobi equations for H_i and $\tilde{H}_i$ have a common solution (cf [12]). Namely, we have the following generalization of the results of [12] to the case of multiparameter generalized Stäckel transform.

Proposition 5. *Suppose that* $\left\{H_{s_i}, H_{s_j}\right\} = 0$ *for all* $i, j = 1, \ldots, k$*. Let* $S = S\big(\boldsymbol{\lambda}, \alpha_1, \ldots, \alpha_k, E_{s_1}, \ldots, E_{s_k}, a_1, \ldots, a_{n-k}\big)$*, where* a_i *are arbitrary constants, be a complete integral of the stationary Hamilton–Jacobi equation for the Hamiltonians* $H_{s_i} = H_{s_i}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \alpha_1, \ldots, \alpha_k)$*,*

$$H_{s_i}(\boldsymbol{\lambda}, \partial S/\partial \boldsymbol{\lambda}, \alpha_1, \ldots, \alpha_k) = E_{s_i}, \qquad i = 1, \ldots, k.$$

If we set $E_{s_i} = \tilde{\alpha}_i$ *and* $\alpha_i = \tilde{E}_{s_i}$ *for all* $i = 1, \ldots, k$ *then* S *is also a complete integral of the stationary Hamilton–Jacobi equation for the Hamiltonians* $\tilde{H}_{s_i} = \tilde{H}_{s_i}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \tilde{\alpha}_1, \ldots \tilde{\alpha}_k)$,

$$\tilde{H}_{s_i}(\boldsymbol{\lambda}, \partial S/\partial \boldsymbol{\lambda}, \tilde{\alpha}_1, \ldots \tilde{\alpha}_k) = \tilde{E}_{s_i}.$$

Further assume that $r \leqslant n$, *and* $\{H_i, H_j\} = 0, i, j = 1, \ldots, r$, *and*

$$S = S(\boldsymbol{\lambda}, \alpha_1, \ldots, \alpha_k, E_1, \ldots, E_r, a_1, \ldots, a_{n-r}) \tag{27}$$

where a_i *are arbitrary constants, is a complete integral for the system of stationary Hamilton–Jacobi equations*

$$H_i(\boldsymbol{\lambda}, \partial S/\partial \boldsymbol{\lambda}, \alpha_1, \ldots, \alpha_k) = E_i, \qquad i = 1, \ldots, r.$$

If we set

$$\alpha_j = \tilde{E}_{s_j}, \qquad E_{s_j} = \tilde{\alpha}_j, \qquad j = 1, \ldots, k, \qquad \text{and} \qquad E_i = \tilde{E}_i,$$
$$i = 1, \ldots, r, \qquad i \neq s_p \qquad \text{for all} \quad p = 1, \ldots, k,$$

then S *(27) is also a complete integral for the system*

$$\tilde{H}_i(\boldsymbol{\lambda}, \partial S/\partial \boldsymbol{\lambda}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) = \tilde{E}_i, \qquad i = 1, \ldots, r.$$

This result suggests that the multiparametric generalized Stäckel transform potentially is a very powerful tool for solving the Hamilton–Jacobi equations (and hence the equations of motion) for Hamiltonian dynamical systems. Indeed, if we can solve the stationary Hamilton–Jacobi equations for the original Hamiltonians H_i, then by proposition 5 we can do this for the transformed Hamiltonians $\tilde{H}_i$ as well, and vice versa.

As for the equations of motion, in addition to general propositions 3 and 4, a somewhat more explicit result can be obtained by straightforward computation.

Corollary 1. *Suppose that* $r = n$, $\{H_i, H_j\} = 0$ *for all* $i, j = 1, \ldots, n$, $\partial^2 H_i/\partial\alpha_j \partial\boldsymbol{\mu} = 0$ *for all* $i = 1, \ldots, n$ *and all* $j = 1, \ldots, k$, *and that* λ_j, $j = 1, \ldots, n$, *can be chosen as local coordinates on the Lagrangian submanifold* $N_E = \{(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in M | H_i(\boldsymbol{\lambda}, \boldsymbol{\mu}, \alpha_1, \ldots, \alpha_k) = E_i, i = 1, \ldots, n\}$ *(in other words, the system* $H_i(\boldsymbol{\lambda}, \boldsymbol{\mu}, \alpha_1, \ldots, \alpha_k) = E_i, i = 1, \ldots, n$, *can be solved for* $\boldsymbol{\mu}$*), and that we have*

$$\begin{aligned} &\alpha_j = \tilde{E}_{s_j}, \qquad E_{s_j} = \tilde{\alpha}_j, \qquad j = 1, \ldots, k, \qquad \text{and} \qquad E_i = \tilde{E}_i, \\ &i = 1, \ldots, n, \qquad i \neq s_p \qquad \text{for all} \quad p = 1, \ldots, k. \end{aligned} \tag{28}$$

Then the reciprocal transformation (26) turns the system

$$\mathrm{d}\boldsymbol{\lambda}/\mathrm{d}t_i = (\partial H_i/\partial \boldsymbol{\mu})|_{N_E}, \qquad i = 1, \ldots, n, \tag{29}$$

into

$$\mathrm{d}\boldsymbol{\lambda}/\mathrm{d}\tilde{t}_i = (\partial \tilde{H}_i/\partial \boldsymbol{\mu})|_{\tilde{N}_{\tilde{E}}}, \qquad i = 1, \ldots, n, \tag{30}$$

where $\tilde{N}_{\tilde{E}} = \{(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in M | \tilde{H}_i(\boldsymbol{\lambda}, \boldsymbol{\mu}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) = \tilde{E}_i, i = 1, \ldots, n\}$.

Recall that N_E and $N_{\tilde{E}}$ in fact represent *the same* Lagrangian submanifold of M, cf the remark after proposition 3.

For instance, if we have $k = 1$, $\alpha_1 \equiv \alpha$, $s_1 = s$, and take

$$H_i = \tfrac{1}{2}(\boldsymbol{\mu}, G_i(\boldsymbol{\lambda})\boldsymbol{\mu}) + V_i(\boldsymbol{\lambda}) + \alpha W_i(\boldsymbol{\lambda}), \qquad i = 1, \ldots, n, \tag{31}$$

where $(\cdot, \cdot)$ stands for the standard scalar product in $\mathbb{R}^n$ and $G_i(\boldsymbol{\lambda})$ are $n \times n$ matrices, then the system (29) reads

$$\mathrm{d}\boldsymbol{\lambda}/\mathrm{d}t_i = G_i(\boldsymbol{\lambda})\boldsymbol{M}, \tag{32}$$

where $\boldsymbol{\mu} = \boldsymbol{M}(\boldsymbol{\lambda}, \alpha, E_1, \ldots, E_n)$ is a general solution of the system $H_i(\alpha, \boldsymbol{\lambda}, \boldsymbol{\mu}) = E_i, i = 1, \ldots, n$.

If we eliminate $\boldsymbol{M}$ from (32) then we obtain the *dispersionless Killing systems* (cf [5, 8, 16, 17])

$$\boldsymbol{\lambda}_{t_i} = G_i(G_s)^{-1}\boldsymbol{\lambda}_{t_s}, \qquad i = 1, \ldots, n, \qquad i \neq s, \tag{33}$$

and the reciprocal transformation (26), which in our case reads

$$\mathrm{d}\tilde{t}_s = -\sum_{i=1}^{n} W_i(\boldsymbol{\lambda})\,\mathrm{d}t_i, \qquad \tilde{t}_i = t_i, \qquad i \neq s,$$

turns (33) into

$$\boldsymbol{\lambda}_{\tilde{t}_i} = \tilde{G}_i(\tilde{G}_s)^{-1}\boldsymbol{\lambda}_{\tilde{t}_s}, \qquad i = 1, \ldots, n, \qquad i \neq s, \tag{34}$$

where the quantities $\tilde{G}_s = -G_s/W_s$ and $\tilde{G}_i = G_i - W_i G_s/W_s, i = 1, 2, \ldots, s-1, s+1, \ldots, n$, are related to the Hamiltonians

$$\tilde{H}_i = \tfrac{1}{2}(\boldsymbol{\mu}, \tilde{G}_i(\boldsymbol{\lambda})\boldsymbol{\mu}) + \tilde{V}_i(\boldsymbol{\lambda}) + \tilde{\alpha}\tilde{W}_i(\boldsymbol{\lambda}), \qquad i = 1, \ldots, n, \tag{35}$$

which are Stäckel-equivalent to $H_i, i = 1, \ldots, n$.

We can now apply proposition 5 in order to obtain the solutions of equations of motion (29) and (30) as follows.

Corollary 2. *Under the assumptions of corollary 1, suppose that*

$$S = S(\boldsymbol{\lambda}, \alpha_1, \ldots, \alpha_k, E_1, \ldots, E_n) \tag{36}$$

is a complete integral for the system of stationary Hamilton–Jacobi equations

$$H_i(\boldsymbol{\lambda}, \partial S/\partial\boldsymbol{\lambda}, \alpha_1, \ldots, \alpha_k) = E_i, \qquad i = 1, \ldots, n.$$

Then a general solution of (29) for $i = d$ can be written in implicit form as

$$\partial S/\partial E_j = \delta_{jd}t_d + b_j, \qquad j = 1, \ldots, n, \tag{37}$$

where b_j are arbitrary constants, and by virtue of (28) a general solution of (30) for $i = d$ can be written in implicit form as

$$\partial S/\partial \tilde{E}_j = \delta_{jd}\tilde{t}_d + b_j, \qquad j = 1, \ldots, n. \tag{38}$$

Comparing (37) and (38) and using (28) we readily see that, in perfect agreement with (26), $t_i = \tilde{t}_i$ for $i \neq s_1, \ldots, s_k$, but $t_{s_j} = \partial S/\partial E_{s_j} - b_{s_j} = \partial S/\partial\tilde{\alpha}_j - b_{s_j}$ while $\tilde{t}_{s_j} = \partial S/\partial\tilde{E}_{s_j} - b_{s_j} = \partial S/\partial\alpha_j - b_{s_j}$. Thus, the above approach does not yield an *explicit* formula expressing $\tilde{t}_{s_j}$ as functions of $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and t_{s_i}.

In order to find a complete integral (36) we can use separation of variables as follows (see e.g. [7, 34] and references therein). Under the assumptions of corollary 2 suppose that $\lambda_i, \mu_i, i = 1, \ldots, n$, are separation coordinates for the Hamiltonians $H_i, i = 1, \ldots, n$, that is, the system of equations $H_i(\boldsymbol{\lambda}, \boldsymbol{\mu}, \alpha_1, \ldots, \alpha_k) = E_i, i = 1, \ldots, n$, is equivalent to the following one:

$$\varphi_i(\lambda_i, \mu_i, \alpha_1, \ldots, \alpha_k, E_1, \ldots, E_n) = 0, \qquad i = 1, \ldots, n, \tag{39}$$

which is nothing but the set of the separation relations[3] on the Lagrangian submanifold N_E.

On the other hand, under the identification (28) the system (39) is equivalent to

$$\tilde{H}_i(\boldsymbol{\lambda}, \boldsymbol{\mu}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) = \tilde{E}_i, \qquad i = 1, \ldots, n. \tag{40}$$

[3] Note that the separation relations involving parameters appear, in a rather different context, in the paper [38] where they are employed for the construction of separation variables.

Thus, the Stäckel-equivalent n-tuples of Hamiltonians share the separation relations (39) provided (28) holds.

Consider the system of stationary Hamilton–Jacobi equations for H_i

$$H_i(\boldsymbol{\lambda}, \partial S/\partial\boldsymbol{\lambda}, \alpha_1, \ldots, \alpha_k) = E_i, \qquad i = 1, \ldots, n. \tag{41}$$

From the above, (41) is equivalent to the system

$$\varphi_i(\lambda_i, \partial S/\partial\lambda_i, \alpha_1, \ldots, \alpha_k, E_1, \ldots, E_n) = 0, \qquad i = 1, \ldots, n. \tag{42}$$

Suppose that (39) can be solved for $\mu_i, i = 1, \ldots, n$:

$$\mu_i = M_i(\lambda_i, \alpha_1, \ldots, \alpha_k, E_1, \ldots, E_n), \qquad i = 1, \ldots, n.$$

Then there exists a separated complete integral of (42), and hence of (41), of the form (cf e.g. [7])

$$S = \sum_{l=1}^{n} \int M_l(\lambda_l, \alpha_1, \ldots, \alpha_k, E_1, \ldots, E_n)\, \mathrm{d}\lambda_l, \tag{43}$$

and general solutions for (29) and (30) can be found using the method of corollary 2.

In this case formulae (37) take the form

$$\sum_{i=1}^{n} \int (\partial M_i(\lambda_i, \alpha_1, \ldots, \alpha_k, E_1, \ldots, E_n)/\partial E_j)\, \mathrm{d}\lambda_i = \delta_{jd} t_d + b_j, \qquad j = 1, \ldots, n, \tag{44}$$

and expressing λ_i as functions of t_d from (44) is nothing but an instance of the Jacobi inversion problem.

6. Multiparameter generalized Stäckel transform and deformations of separation curves

Under the assumptions of corollary 1, suppose that $\lambda_i, \mu_i, i = 1, \ldots, n$, are *separation coordinates* for the n-tuple of commuting Hamiltonians $H_i, i = 1, \ldots, n$. Then the Lagrangian submanifold N_E is defined by n separation relations (39). Further assume that all functions φ_i are identical:

$$\varphi_i = \varphi(\lambda_i, \mu_i, \alpha_1, \ldots, \alpha_k, E_1, \ldots, E_n), \qquad i = 1, \ldots, n. \tag{45}$$

Then relations (39) mean that the points $(\lambda_i, \mu_i), i = 1, \ldots, n$, belong to the *separation curve* [7, 34]

$$\varphi(\lambda, \mu, \alpha_1, \ldots, \alpha_k, E_1, \ldots, E_n) = 0. \tag{46}$$

If the relations

$$\varphi(\lambda_i, \mu_i, \alpha_1, \ldots, \alpha_k, H_1, \ldots, H_n) = 0, \qquad i = 1, \ldots, n,$$

uniquely determine the Hamiltonians H_i for $i = 1, \ldots, n$, then for the sake of brevity we shall say that H_i for $i = 1, \ldots, n$ have the *separation curve*

$$\varphi(\lambda, \mu, \alpha_1, \ldots, \alpha_k, H_1, \ldots, H_n) = 0. \tag{47}$$

Fixing values of all Hamiltonians $H_i = E_i, i = 1, \ldots, n$, picks a particular Lagrangian submanifold from the Lagrangian foliation. It is also clear that the Stäckel-equivalent n-tuples of the Hamiltonians $H_i, i = 1, \ldots, n$, and $\tilde{H}_i, i = 1, \ldots, n$, share the separation curve (47) provided (3) and (5) hold.

In the rest of this section we shall deal with a special class of separation curves of the form (cf e.g. [7] and references therein)

$$\sum_{j=1}^{n} H_j \lambda^{\beta_j} = \psi(\lambda, \mu), \tag{48}$$

where β_j are arbitrary pairwise distinct non-negative integers, $\beta_1 > \beta_2 > \cdots > \beta_n$. In fact one can always impose the normalization $\beta_n = 0$ by dividing the left- and right-hand side of (48) by λ^{β_n} if necessary, but we shall not impose this normalization in the present paper.

For a given n, each class of systems (48) is labeled by a sequence $(\beta_1, \ldots, \beta_n)$ while a particular system from a class is given by a particular choice of $\psi(\lambda, \mu)$. In particular, the choice $\psi(\lambda, \mu) = \frac{1}{2} f(\lambda)\mu^2 + \gamma(\lambda)$ yields the well-known classical Stäckel systems. All these systems admit the separation of variables in the same coordinates (λ_i, μ_i) by construction.

We shall refer to the class with the separation curve

$$\sum_{j=1}^{n} H_j \lambda^{n-j} = \psi(\lambda, \mu) \tag{49}$$

as to the *seed class*. Note that if $\psi(\lambda, \mu) = \frac{1}{2} f(\lambda)\mu^2 + \gamma(\lambda)$ we obtain precisely the Benenti class of Stäckel systems [1, 2]. The seed class is a rather general one: it includes the majority of known integrable systems with natural Hamiltonians [7].

It turns out that, roughly speaking, the n-tuple of Hamiltonians having the general separation curve (48) can be related via a suitably chosen generalized multiparameter Stäckel transform to an n-tuple of Hamiltonians having the separation curve (49) from the seed class. The exact picture is a bit more involved, as in fact we need to consider the *deformations* of the curves in question.

Define first an operator R_k^f that acts as follows:

$$R_k^f(F) = F + f\lambda^k - (\lambda^k/k!)(\partial^k F/\partial\lambda^k)|_{\lambda=0}.$$

For instance, we have

$$R_k^f\left(\sum_{j=0}^{s} a_j\lambda^j\right) = f\lambda^k + \sum_{j=0, j\neq k}^{s} a_j\lambda^j.$$

Now let

$$F_0 = \sum_{j=1}^{n} H_j\lambda^{n-j} \qquad \text{and} \qquad \tilde{F}_0 = \sum_{j=1}^{n} \tilde{H}_j\lambda^{n-j}.$$

For any integer m define [7] the so-called basic separable potentials $V_j^{(m)}$ by means of the relations

$$\lambda^m + \sum_{j=1}^{n} V_j^{(m)}\lambda^{n-j} = 0 \tag{50}$$

that must hold for $\lambda = \lambda_i$, $i = 1, \ldots, n$.

Under the assumptions of corollary 1, consider an n-tuple of commuting Hamiltonians of the form

$$H_i = H_i^{(0)} + \sum_{j=1}^{k} \alpha_j V_i^{(\gamma_j)}, \tag{51}$$

where γ_j, $j = 1, \ldots, k$, are pairwise distinct integers.

Suppose that the Hamiltonians (51) have the separation curve of the form

$$\sum_{j=1}^{k} \alpha_j \lambda^{\gamma_j} + F_0 = \psi(\lambda, \mu), \tag{52}$$

where $\gamma_j > n - 1$ for all $j = 1, \ldots, k$, and $\gamma_i \neq \gamma_j$ if $i \neq j$ for all $i, j = 1, \ldots, k$.

Now pick $k \leqslant n$ distinct numbers $s_i \in \{1, \ldots, n\}$ and define the Hamiltonians $\tilde{H}_i$ by means of the following separation curve:

$$\sum_{j=1}^{k} \tilde{H}_{s_j} \lambda^{\gamma_j} + R_{n-s_1}^{\tilde{\alpha}_1} \cdots R_{n-s_k}^{\tilde{\alpha}_k}(\tilde{F}_0) = \psi(\lambda, \mu). \tag{53}$$

This means that $\tilde{H}_i$ are the solutions of the system of linear algebraic equations obtained from (53) upon substituting λ_i for λ and μ_i for μ into (53) for $i = 1, \ldots, n$.

Proposition 6. *Under the above assumptions the n-tuple of Hamiltonians $\tilde{H}_i, i = 1, \ldots, n$, is Stäckel-equivalent to $H_i, i = 1, \ldots, n$.*

The k-parameter generalized Stäckel transform relating $\tilde{H}_i, i = 1, \ldots, n$ to $H_i, i = 1, \ldots, n$ reads as follows:

$$\tilde{H}_{s_i} = \det B_i / \det B, \tag{54}$$

where

$$B = \begin{Vmatrix} V_{s_1}^{(\gamma_1)} & \cdots & V_{s_1}^{(\gamma_k)} \\ \vdots & \ddots & \vdots \\ V_{s_k}^{(\gamma_1)} & \cdots & V_{s_k}^{(\gamma_k)} \end{Vmatrix}$$

is a $k \times k$ matrix, and B_i are obtained from B by replacing $V_{s_j}^{(\gamma_i)}$ by $\tilde{\alpha}_j - H_{s_j}^{(0)}$ for all $j = 1, \ldots, k$;

$$\tilde{H}_i = H_i^{(0)} + \sum_{j=1}^{k} \tilde{H}_{s_j} V_i^{(\gamma_j)}, \qquad i = 1, \ldots, r, \qquad i \neq s_j \qquad \text{for} \quad j = 1, \ldots, k, \tag{55}$$

where $\tilde{H}_{s_i}$ are given by (54).

Proof. First, note that the above formulae for $\tilde{H}_i$ indeed constitute the Stäckel transform, as equation (54) is readily seen to imply the relations of the type (3), namely

$$H_{s_i}^{(0)} + \sum_{j=1}^{k} \tilde{H}_{s_j} V_{s_i}^{(\gamma_j)} = \tilde{\alpha}_i, \qquad i = 1, \ldots, k, \tag{56}$$

cf the discussion after (8).

Now we only have to prove that the Hamiltonians $\tilde{H}_i$ defined by (54) and (55) have the separation curve (53). As we have already mentioned above, the Stäckel-equivalent n-tuples of separable commuting Hamiltonians share the separation relations provided (28) holds. Therefore, in order to prove our claim it suffices to show that the separation curves (52) and (53) can be identified by virtue of (56).

Indeed, upon plugging into (52) the relations

$$\lambda^{\gamma_j} = -\sum_{p=1}^{n} V_p^{(\gamma_j)} \lambda^{n-p}, \qquad j = 1, \ldots, k, \tag{57}$$

that follow from (50), collecting the coefficients at the powers of λ, and taking into account (51), the separation curve (52) can be rewritten as

$$\sum_{j=1}^{n} H_j^{(0)} \lambda^{n-j} = \psi(\lambda, \mu). \tag{58}$$

On the other hand, plugging (57) into (53) and proceeding in a similar fashion as above, we obtain

$$-\sum_{p=1}^{n} \left(\sum_{j=1}^{k} \tilde{H}_{s_j} V_p^{(\gamma_j)} \right) \lambda^{n-p} + R_{n-s_1}^{\tilde{\alpha}_1} \cdots R_{n-s_k}^{\tilde{\alpha}_k} (\tilde{F}_0) = \psi(\lambda, \mu). \tag{59}$$

By virtue of relations (56), which can be further rewritten as

$$H_{s_i}^{(0)} = -\sum_{j=1}^{k} \tilde{H}_{s_j} V_i^{(\gamma_j)} + \tilde{\alpha}_i, \qquad i = 1, \ldots, k,$$

along with (55), we find that the curves (59) and (58) are indeed identical, and hence so are the curves (53) and (52). □

Remark 1. In fact the above argument can be inverted, that is, we can obtain the relations (56) (and hence (54)) and (55) by requiring the curves (52) and (53) to coincide and comparing the coefficients at the powers of λ on the left-hand sides of these curves, or equivalently (by virtue of (50)), of (59) and (58).

Proposition 7. *The inverse of the k-parameter generalized Stäckel transform (54), (55) has the form*

$$H_{s_i} = \det \tilde{B}_i / \det \tilde{B}, \tag{60}$$

where

$$\tilde{B} = \begin{Vmatrix} \tilde{V}_{s_1}^{(n-s_1)} & \cdots & \tilde{V}_{s_1}^{(n-s_k)} \\ \vdots & \ddots & \vdots \\ \tilde{V}_{s_k}^{(n-s_1)} & \cdots & \tilde{V}_{s_k}^{(n-s_k)} \end{Vmatrix}$$

is a $k \times k$ *matrix, and* $\tilde{B}_i$ *are obtained from* $\tilde{B}$ *by replacing* $\tilde{V}_{s_j}^{(n-s_i)}$ *by* $\alpha_j - \tilde{H}_{s_j}^{(0)}$ *for all* $j = 1, \ldots, k$;

$$H_i = \tilde{H}_i^{(0)} + \sum_{j=1}^{k} H_{s_j} \tilde{V}_i^{(n-s_j)}, \qquad i = 1, \ldots, r, \qquad i \neq s_j \quad \text{for} \quad j = 1, \ldots, k, \tag{61}$$

where H_{s_i} *are given by (60) and* $\tilde{V}_j^{(m)}$ *are deformed separable potentials defined for all integer* m *by means of the relations*

$$\lambda^m + \sum_{j=1}^{k} \tilde{V}_{s_j}^{(m)} \lambda^{\gamma_j} + \sum_{p=1,\, p\neq s_1,\ldots,s_k}^{n} \tilde{V}_p^{(m)} \lambda^{n-p} = 0 \tag{62}$$

that must hold for $\lambda = \lambda_i$, $i = 1, \ldots, n$.

The proof of this result is readily obtained from that of proposition 6 using the fact that the inverse of the n-parameter generalized Stäckel transform (54), (55) is nothing but the dual of the latter (see section 2 for the definition of duality).

Notice that upon setting the parameters α_i and $\tilde{\alpha}_i$ to zero for all $i = 1, \ldots, k$ formulae (54) and (55) indeed relate the Hamiltonians H_i with the separation curve (49) and the Hamiltonians

$\tilde{H}_i$ with the separation curve (48). In this case we essentially recover the formulae from [7] relating the Hamiltonians from the seed class and from the so-called k-hole deformation thereof (in our language, the deformed systems are precisely those having the separation curve (48)) up to a suitable renumeration of the Hamiltonians $\tilde{H}_i$.

As a final remark, note that in fact propositions 6 and 7 hold for more general classes of Hamiltonians than those defined via (52) and (53). Namely, the propositions in question remain valid if we pass from the separation curves to the separation relations, i.e., if we define the Hamiltonians H_i by means of the separation relations

$$\sum_{j=1}^{k} \alpha_j \lambda_p^{\gamma_j} + F_0(\lambda_p) = \psi_p(\lambda_p, \mu_p), \qquad p = 1, \ldots, n, \tag{52$'$}$$

instead of (52), and the Hamiltonians $\tilde{H}_i$ by means of the separation relations

$$\sum_{j=1}^{k} \tilde{H}_{s_j} \lambda_p^{\gamma_j} + R_{n-s_1}^{\tilde{\alpha}_1} \cdots R_{n-s_k}^{\tilde{\alpha}_k}(\tilde{F}_0(\lambda_p)) = \psi_p(\lambda_p, \mu_p), \qquad p = 1, \ldots, n, \tag{53$'$}$$

instead of (53). Here $F_0(\lambda) = \sum_{j=1}^{n} H_j \lambda^{n-j}$ and $\tilde{F}_0(\lambda) = \sum_{j=1}^{n} \tilde{H}_j \lambda^{n-j}$. Let us stress that in this new setting the functions $\psi_p(\lambda, \mu)$ with different p are no longer obliged to be identical.

7. Examples

As a simple illustration of the above results, consider the Hamiltonian systems on a four-dimensional phase space $M = \mathbb{R}^4$ with the coordinates (p_1, p_2, q_1, q_2) and canonical Poisson structure.

For our first example let $k = 1, r = 2, s_1 = 2, \alpha_1 \equiv \alpha$ and $\tilde{\alpha}_1 \equiv \tilde{\alpha}$. Consider the Hamiltonian

$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{\alpha\left(q_1^2 - q_2^2\right)}{q_2} p_2 - 2\alpha^2 q_1^2,$$

which is Liouville integrable because it Poisson commutes with

$$H_2 = \frac{q_1 p_2 - q_2 p_1 - 2\alpha q_1 q_2}{p_2}.$$

The above pair of commuting Hamiltonians was found by analogy with one of the models from [20].

Relation (3) in this case takes the form

$$\frac{q_1 p_2 - q_2 p_1 - 2\tilde{H}_2 q_1 q_2}{p_2} = \tilde{\alpha},$$

whence

$$\tilde{H}_2 = \frac{q_1 p_2 - q_2 p_1 - \tilde{\alpha} p_2}{2 q_1 q_2},$$

and therefore by virtue of (4) we have

$$\tilde{H}_1 = \frac{q_1^2 + q_2^2 - 2\tilde{\alpha} q_1}{2 q_1 q_2} p_1 p_2 + \frac{\tilde{\alpha}\left(q_1^2 - \tilde{\alpha} q_1 + q_2^2\right)}{2 q_1 q_2^2} p_2^2.$$

By proposition 1 (ii) the relation $\{H_1, H_2\} = 0$ implies $\{\tilde{H}_1, \tilde{H}_2\} = 0$, so $\tilde{H}_1$ is Liouville integrable just like H_1. Interestingly enough, in this example the generalized Stäckel transform sends the Hamiltonian H_1 into a natural *geodesic* Hamiltonian $\tilde{H}_1$, but the metric associated with $\tilde{H}_1$ is not flat and, moreover, has nonconstant scalar curvature unlike the metric associated with H_1.

By proposition 4 the reciprocal transformation

$$\tilde{t}_1 = t_1, \qquad d\tilde{t}_2 = \left(-2q_1p_1 + \frac{\left(q_1^2 - 2\tilde{\alpha}q_1 + q_2^2\right)p_2}{q_2}\right)dt_1 + \frac{2q_1q_2}{p_2}dt_2$$

takes the equations of motion for H_1 and H_2, with the respective evolution parameters t_1 and t_2, restricted onto the level surface $N_{\tilde{\alpha}} = \{x \in \mathbb{R}^4 | H_2(x, \alpha) = \tilde{\alpha}\}$ into the equations of motion for $\tilde{H}_1$ and $\tilde{H}_2$, with the respective evolution parameters $\tilde{t}_1$ and $\tilde{t}_2$, restricted onto the level surface $\tilde{N}_\alpha = \{x \in \mathbb{R}^4 | \tilde{H}_2(x, \tilde{\alpha}) = \alpha\}$. It is easily seen that $\tilde{N}_\alpha$ and $N_{\tilde{\alpha}}$ indeed represent the same submanifold of $\mathbb{R}^4$.

For the second example consider the (extended) Hénon–Heiles system with the Hamiltonian

$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \alpha_1\left(q_1^3 + \frac{q_1q_2^2}{2}\right) - \alpha_2q_1,$$

which Poisson commutes with

$$H_2 = \frac{1}{2}q_2p_1p_2 - \frac{1}{2}q_1p_2^2 - \alpha_1\left(\frac{q_2^4}{16} + \frac{q_1^2q_2^2}{4}\right) - \alpha_2\frac{q_2^2}{4}.$$

The separation curve for the system in question belongs to the seed class and reads

$$\alpha_1\lambda^4 + \alpha_2\lambda^2 + H_1\lambda + H_2 = \lambda\mu^2/2. \tag{63}$$

The separation coordinates (λ_i, μ_i), $i = 1, 2$, are related to p's and q's by the formulae

$$q_1 = \lambda_1 + \lambda_2, \qquad q_2 = 2\sqrt{-\lambda_1\lambda_2},$$
$$p_1 = \frac{\lambda_1\mu_1}{\lambda_1 - \lambda_2} + \frac{\lambda_2\mu_2}{\lambda_2 - \lambda_1}, \qquad p_2 = \sqrt{-\lambda_1\lambda_2}\left(\frac{\mu_1}{\lambda_1 - \lambda_2} + \frac{\mu_2}{\lambda_2 - \lambda_1}\right).$$

Let $s_1 = 1$, $s_2 = 2$, $k = r = 2$. Then (51) and (54) yield the following deformation of H_1 and H_2:

$$\tilde{H}_1 = \frac{2}{q_1q_2^2}p_1^2 - \frac{8}{q_2^3}p_1p_2 - \frac{2\left(q_2^2 + 4q_1^2\right)}{q_1q_2^4}p_2^2 - \frac{4}{q_1q_2^2}\tilde{\alpha}_1 + \frac{16}{q_2^4}\tilde{\alpha}_2,$$
$$\tilde{H}_2 = -\frac{4q_1^2 + q_2^2}{2q_1q_2^2}p_1^2 - \frac{4\left(q_2^2 + 2q_1^2\right)}{q_2^3}p_1p_2 + \frac{16q_1^4 + 12q_1^2q_2^2 + q_2^4}{q_1q_2^4}p_2^2 + \frac{\left(q_2^2 + 4q_1^2\right)}{q_1q_2^2}\alpha_1 - \frac{8\left(q_2^2 + 2q_1^2\right)}{q_2^4}\alpha_2.$$

The corresponding separation curve reads (see proposition 6)

$$\tilde{H}_1\lambda^4 + \tilde{H}_2\lambda^2 + \tilde{\alpha}_1\lambda + \tilde{\alpha}_2 = \lambda\mu^2/2. \tag{64}$$

Using proposition 3 and proceeding in analogy with the previous example we readily find that the reciprocal transformation (17) for the equations of motion restricted onto the appropriate Lagrangian submanifolds in our case takes the form

$$d\tilde{t}_1 = \left(q_1^3 + \frac{q_1q_2^2}{2}\right)dt_1 + \left(\frac{q_2^4}{16} + \frac{q_1^2q_2^2}{4}\right)dt_2, \qquad d\tilde{t}_2 = q_1^2\,dt_1 + \frac{q_2^2}{4}dt_2.$$

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